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On monotone pointwise contractions in Banach and metric spaces

Afrah AN Abdou[1*] and Mohamed A Khamsi[2,3]

*Correspondence:
aabdou@kau.edu.sa
[1]Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia
Full list of author information is available at the end of the article

Abstract
In this work we define the new concept of monotone pointwise contraction mappings in Banach and metric spaces. Then we prove the existence of fixed points of such mappings.

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Keywords: fixed point; CAT(0) spaces; hyperbolic metric spaces; monotone mapping; pointwise contraction

1 Introduction

The notion of asymptotic pointwise mappings was introduced in [1–3]. The ultrapower technique was useful in proving some related fixed point results. In [3], the authors gave simple and elementary proofs for the existence of fixed point theorems for asymptotic pointwise mappings without the use of ultrapowers. In [4], most of these results were extended to metric spaces. In this work, we introduce the new concept of monotone mappings in Banach and metric spaces. Indeed recently a new direction has been discovered dealing with the extension of the Banach contraction principle to metric spaces endowed with a partial order. The first attempt was successfully carried out by Ran and Reurings [5]. In particular, they showed how this extension is useful when dealing with some special matrix equations. Another similar approach was carried out by Nieto and Rodríguez-López [6] and they used such arguments in solving some differential equations. In [7], Jachymski gave a more general unified version of these extensions by considering graphs instead of a partial order.

In this work, we investigate the fixed point theory of pointwise contraction mappings to the case of monotone mappings. In particular, we will extend the main result of [3] to the case of monotone mappings. Our approach is new and different from the ideas found in [5, 6].

For more on metric fixed point theory, the reader may consult the book [8].

2 Preliminaries

Let $(X, \|\cdot\|)$ be a Banach vector space and suppose $\leq$ is a partial order on X. As usual we adopt the convention $x \geq y$ if and only if $y \leq x$. The linear structure of X is assumed to be compatible with the order structure in the following sense:

(i) $x \leq y \Rightarrow x + z \leq y + z$ for all $x, y, z \in X$;
(ii) $x \leq y \Rightarrow \alpha x \leq \alpha y$ for all $x, y \in X$ and $\alpha \in \mathbb{R}^+$.

Moreover, we will assume that order intervals are closed. Recall that an order interval is any of the subsets:

(i) $[a, \rightarrow) = \{x \in X; a \preceq x\}$,

(ii) $(\leftarrow, a] = \{x \in X; x \preceq a\}$,

for any $a \in X$. Next we give the definition of monotone Lipschitzian mappings.

Definition 1 Let $(X, \|\cdot\|, \leq)$ be as above. Let C be a nonempty subset of X. A map $T : C \to X$ is said to be:

(a) monotone if $T(x) \leq T(y)$ whenever $x \leq y$,

(b) monotone pointwise Lipschitzian, if T is monotone and for any $x \in X$, there exists $k(x) \in [0, +\infty)$ such that

$$\|T(x) - T(y)\| \leq k(x)\|x - y\|,$$

for any $y \in X$ such that x and y are comparable, *i.e.*, $x \leq y$ or $y \leq x$.

If $k(x) < 1$, for any $x \in X$, then T is said to be a monotone pointwise contraction. A fixed point of T is any element $x \in C$ such that $T(x) = x$. The set of all fixed points of T is denoted by $\operatorname{Fix}(T)$.

It is clear that pointwise contractive behavior was introduced to extend the contractive behavior in Banach contraction principle [9].

Example 1 Let K be a bounded closed convex subset of the Hilbert space ℓ^2. Let $F : K \to K$. In [9] it is shown that if F is defined and continuously Fréchet differentiable on a convex open set containing K, then F is a pointwise contraction on K if and only if $\|F'_x\| < 1$, for each $x \in K$, where F'_x denotes the Fréchet derivative of F at x. Next consider the metric space

$$M = \{0,1\} \times K = \{(0,x); x \in K\} \cup \{(1,x); x \in K\}.$$

The distance d on M is defined by

$$d\big((\varepsilon_1, x_1), (\varepsilon_2, x_2)\big) = |\varepsilon_1 - \varepsilon_2| + \|x_1 - x_2\|.$$

The partial order $\preceq$ on M is defined by:

(i) $(0,x)$ and $(1,y)$ are not comparable for any $x, y \in K$;

(ii) $(\varepsilon, x) \preceq (\varepsilon, y)$ if and only if $x \leq y$ (using the natural pointwise order in ℓ^2), for any $\varepsilon \in \{0,1\}$ and $x, y \in K$.

Define the mapping $T : M \to M$ by

$$T\big((\varepsilon, x)\big) = \big(1 - \varepsilon, F(x)\big),$$

where $F : K \to K$ is continuously Fréchet differentiable on a convex open set containing K such that $\|F'_x\| < 1$, for each $x \in K$. We also assume that F is monotone, *i.e.*, $x \leq y$ implies $F(x) \leq F(y)$ for any $x, y \in K$. Then T is monotone pointwise contraction on M. Indeed, any two points of M are comparable if and only if they have the same first component. Next we notice

$$\begin{aligned} d\big(T\big((\varepsilon,x)\big),T\big((\varepsilon,y)\big)\big) &= d\big(\big(1-\varepsilon,F(x)\big),\big(1-\varepsilon,F(y)\big)\big) \\ &= \big\|F(x)-F(y)\big\| \\ &\leq \alpha(x)\|x-y\| \\ &= \alpha(x)d\big((\varepsilon,x),(\varepsilon,y)\big), \end{aligned}$$

where $\alpha(x) \in [0,1)$, for any $\varepsilon \in \{0,1\}$ and $x,y \in K$ comparable. Clearly we used the fact that F is a pointwise contraction on K. But T is not a pointwise contraction on M since

$$d\big(T\big((0,x)\big),T\big((1,x)\big)\big) = d\big((0,x),(1,x)\big) = 1,$$

for any $x \in K$.

The central fixed point result for pointwise contraction mappings is the following theorem [1, 2].

Theorem 2.1 *Let K be a weakly compact convex subset of a Banach space and suppose $T : K \to K$ is a pointwise contraction. Then T has a unique fixed point, x_0. Moreover, the orbit $\{T^n(x)\}$ converges to x_0, for each $x \in M$.*

Note that if T is a monotone pointwise contraction, then it is not necessarily continuous by contrast to the pointwise contraction case. Since the main focus of this paper is the fixed point problem, we have the following result.

Lemma 2.1 *Let $(X,\|\cdot\|,\leq)$ be as above. Let C be a nonempty subset of X. Let $T : C \to X$ be a monotone pointwise contraction. If $a \in \mathrm{Fix}(T)$, then for any $x \in X$ comparable to a, i.e., $a \leq x$ or $x \leq a$, we have $\{T^n(x)\}$ converges to a. In particular, if a and b are two comparable fixed points of T, then we must have $a = b$.*

Proof Let $a \in \mathrm{Fix}(T)$ and x comparable to a. Assume that $a \leq x$. Since T is monotone, we have $a \leq T^n(x)$, for any $n \geq 1$. Using the definition of pointwise contraction we get

$$\big\|T^n(x)-a\big\| \leq k(a)\big\|T^{n-1}(x)-a\big\| \leq k(a)^n\|x-a\|,$$

for any $n \geq 1$. Since $k(a) < 1$, we conclude that $\{T^n(a)\}$ converges to a. Obviously if a and b are two fixed points of T and $a \leq b$, then we have $\{T^n(b)\} = \{b\}$ converges to a, which implies $a = b$. □

The crucial part in dealing with pointwise contractions is the existence of the fixed point. Usually it takes more assumptions than the classical Banach contraction principle.

3 Existence of fixed point of monotone pointwise contractions in Banach spaces

Let $(X,\|\cdot\|,\leq)$ be as above. Let C be a nonempty convex subset of X. Let $T : C \to C$ be a monotone mapping. Assume there exists $x_1 \in C$ such that $x_1 \leq T(x_1)$. Fix $\lambda \in (0,1)$.

Consider the Krasnoselskii iteration sequence $\{x_n\} \subset C$ defined by

$$x_{n+1} = \lambda x_n + (1-\lambda)T(x_n), \quad n \geq 1. \tag{1}$$

Since order intervals are convex, we have $x_1 \leq x_2 \leq T(x_1)$. Since T is monotone, we get $T(x_1) \leq T(x_2)$. By induction, we will prove that

$$x_n \leq x_{n+1} \leq T(x_n) \leq T(x_{n+1}),$$

for any $n \geq 1$. Let ω be a weak-cluster point of $\{x_n\}$, *i.e.*, there exists a subsequence $\{x_{\phi(n)}\}$ which converges weakly to ω. Since order intervals are closed and convex, we conclude that $x_n \leq \omega$, for any $n \geq 1$, since $\{x_n\}$ is a monotone increasing sequence. From this we conclude that $\{x_n\}$ has at most one weak-cluster point.

Theorem 3.1 *Let $(X, \|\cdot\|, \leq)$ be as above. Let C be a weakly compact nonempty convex subset of X. Let $T : C \to C$ be a monotone pointwise contraction. Assume there exists $x_1 \in C$ such that $x_1 \leq T(x_1)$. Consider the sequence $\{x_n\}$ defined by* (1). *Then $\{x_n\}$ is weakly convergent and* $\lim_{n\to\infty} \|x_n - T(x_n)\| = 0$.

Proof Since C is weakly compact, the sequence $\{x_n\}$ has at least one weak-cluster point. Since $\{x_n\}$ has at most one weak-cluster point, we conclude that $\{x_n\}$ is weakly convergent. As for the second part of this theorem, we will need the following inequality, proved in [10, 11]:

$$\begin{aligned}(1+n\lambda)\left\|T(x_i)-x_i\right\| \leq{}& \left\|T(x_{i+n})-x_i\right\| \\ &+ (1-\lambda)^{n-1}\left(\left\|T(x_i)-x_i\right\| - \left\|T(x_{i+n})-x_{i+n}\right\|\right),\end{aligned} \tag{2}$$

for any $i, n \in \mathbb{N}$, which is proved by induction on i. Next we note first that $\{\|x_n - T(x_n)\|\}$ is decreasing. Indeed we have $x_{n+1} - x_n = (1-\lambda)(T(x_n) - x_n)$, for any $n \geq 0$. Therefore $\{\|x_n - T(x_n)\|\}$ is decreasing if and only if $\{\|x_{n+1} - x_n\|\}$ is decreasing, which holds since

$$\|x_{n+2} - x_{n+1}\| \leq \lambda\|x_{n+1} - x_n\| + (1-\lambda)\left\|T(x_{n+1}) - T(x_n)\right\| \leq \|x_{n+1} - x_n\|,$$

for any $n \geq 0$. So if we assume that $\lim_{n\to+\infty} \|x_n - T(x_n)\| = R > 0$, then we let $i \to +\infty$ in the inequality (2) to obtain

$$(1+n\lambda)R \leq \delta(C),$$

for any $n \in \mathbb{N}$, where $\delta(C) = \operatorname{diam}(C)$. Obviously this is a contradiction since both λ and R are not equal to 0. Therefore we have

$$\lim_{n\to+\infty} \left\|x_n - T(x_n)\right\| = 0. \qquad \square$$

Remark 1 Let ω be the weak limit of $\{x_n\}$. Using the properties of the order intervals, we conclude that $x_n \leq \omega$, for any $n \geq 1$. The monotonicity of T implies $T(x_n) \leq T(\omega)$, for any $n \geq 1$. Since $\{x_n\}$ is a quasi-fixed point sequence of T, *i.e.*, $\lim_{n\to+\infty} \|x_n - T(x_n)\| = 0$, we conclude that $\{T(x_n)\}$ also weakly converges to ω, which implies $\omega \leq T(\omega)$.

Next we prove the main result of this work, which can be seen as an analog to Theorem 2.1.

Theorem 3.2 *Let $(X, \|\cdot\|, \leq)$ be as above. Let C be a weakly compact nonempty convex subset of X. Let $T : C \to C$ be a monotone pointwise contraction. Assume there exists $x_1 \in C$ such that $x_1 \leq T(x_1)$. Then T has a fixed point $z \in C$ and $\{T^n(x_1)\}$ converges to z.*

Proof Let $\{x_n\}$ be the sequence generated from x_1 by (1). Let ω be its weak limit. Define the type function $\tau : C \to [0, +\infty)$ by

$$\tau(x) = \limsup_{n\to\infty} \|x_n - x\|.$$

Set $\tau_0 = \inf\{\tau(x); x \geq \omega\}$. Obviously τ is a continuous convex function. Since C is weakly compact convex subset of X, we conclude the existence of $z \in C$ such that $z \geq \omega$ and $\tau(z) = \tau_0$. Since T is monotone, we have $T(\omega) \leq T(z)$, which implies $\omega \leq T(z)$ by using Remark 1. Therefore we must have $\tau(z) \leq \tau(T(z))$. On the other hand, we have

$$\tau\bigl(T(z)\bigr) = \limsup_{n\to\infty} \bigl\|x_n - T(z)\bigr\| = \limsup_{n\to\infty} \bigl\|T(x_n) - T(z)\bigr\|.$$

Since $x_n \leq \omega \leq z$, for any $n \geq 1$, we get

$$\bigl\|T(x_n) - T(z)\bigr\| \leq k(z)\|x_n - z\|, \quad n \geq 1,$$

which implies

$$\limsup_{n\to\infty} \bigl\|T(x_n) - T(z)\bigr\| \leq k(z) \limsup_{n\to\infty} \|x_n - z\|.$$

Hence $\tau(z) \leq k(z)\tau(z)$. Since $k(z) < 1$, we conclude that $\tau(z) = 0$, which implies

$$\limsup_{n\to\infty} \|x_n - z\| = 0,$$

i.e., $\{x_n\}$ converges to z. In other words, we have $\omega = z$. Next we show that $z \in \mathrm{Fix}(T)$. Since

$$\bigl\|T(x_n) - T(z)\bigr\| \leq k(z)\|x_n - z\|, \quad n \geq 1,$$

we conclude that $\{T(x_n)\}$ converges to $T(z)$. Finally $\lim_{n\to\infty} \|x_n - T(x_n)\| = 0$ implies that $T(z) = z$. The last part of this theorem follows from the fact that $x_1 \leq z$. □

Note that the conclusion of Theorem 3.2 is still valid if we assume $T(x_1) \leq x_1$. In this case the Krasnoselskii sequence $\{x_n\}$ will be monotone decreasing and its limit will be less than x_1.

4 Existence of fixed point of monotone pointwise contractions in metric spaces

In this section, we discuss the analog of Theorem 3.2 in metric spaces as the authors of [4] did. The approach of both the linear and the nonlinear cases uses an intersection property of convex sets [3] of admissible sets [4]. Since the proof of the main result of the previous section is based on the Krasnoselskii iteration, we will need some kind of convex combination. This is the reason why we do need some kind of metric convexity.

Let (X,d) be a metric space. Suppose that there exists a family $\mathcal{F}$ of metric segments such that any two points x, y in X are endpoints of a unique metric segment $[x,y] \in \mathcal{F}$ ($[x,y]$ is an isometric image of the real line interval $[0,d(x,y)]$). We shall denote by $\beta x \oplus (1-\beta)y$ the unique point z of $[x,y]$ which satisfies

$$d(x,z) = (1-\beta)d(x,y) \quad \text{and} \quad d(z,y) = \beta d(x,y),$$

where $\beta \in [0,1]$. Such metric spaces are usually called *convex metric spaces* [12]. Moreover, if we have

$$d\big(\alpha p \oplus (1-\alpha)x, \alpha q \oplus (1-\alpha)y\big) \leq \alpha d(p,q) + (1-\alpha)d(x,y),$$

for all p, q, x, y in X, and $\alpha \in [0,1]$, then X is said to be a *hyperbolic metric space* (see [13]). Throughout this paper, we will assume

$$\alpha x \oplus (1-\alpha)y = (1-\alpha)y \oplus \alpha x,$$

for any $\alpha \in [0,1]$, and any $x, y \in X$.

Obviously, normed linear spaces are hyperbolic spaces. As nonlinear examples, one can consider the Hadamard manifolds [14], the Hilbert open unit ball equipped with the hyperbolic metric [15], and the CAT(0) spaces [16–18]. We will say that a subset C of a hyperbolic metric space X is convex if $[x,y] \subset C$ whenever x, y are in C.

Definition 2 Let (X,d) be a hyperbolic metric space. Let C be a nonempty convex closed subset of X. We say that C is weakly compact if and only if for any decreasing sequence $\{C_n\}$ of closed nonempty convex subsets of C, we see that $\bigcap_{n\geq 1} C_n$ is not empty.

Since in this work we discuss the fixed point theory of monotone mappings, we will need to introduce a partial order in (X,d). Indeed we assume that a partial order $\preceq$ exists in X. As usual we adopt the convention $x \succeq y$ if and only if $y \preceq x$. Throughout we assume that order intervals are closed and convex. Recall that an order interval is any of the subsets:

(i) $[a,\rightarrow) = \{x \in X; a \preceq x\}$,
(ii) $(\leftarrow,a] = \{x \in X; x \preceq a\}$,

for any $a \in X$. Next we give the definition of monotone Lipschitzian mappings.

Definition 3 Let $(X,d,\preceq)$ be as above. Let C be a nonempty subset of X. A map $T: C \to X$ is said to be:

(a) monotone if $T(x) \preceq T(y)$ whenever $x \preceq y$,
(b) monotone pointwise Lipschitzian, if T is monotone and for any $x \in X$, there exists $k(x) \in [0,+\infty)$ such that

$$d\big(T(x), T(y)\big) \leq k(x)d(x,y),$$

for any $y \in X$ such that x and y are comparable.

If $k(x) < 1$, for any $x \in X$, then T is said to be a monotone pointwise contraction.

Let C be a nonempty convex subset of X. Let $T : C \to C$ be a monotone mapping. Assume there exists $x_1 \in C$ such that $x_1 \leq T(x_1)$. Fix $\lambda \in (0,1)$. Consider the Krasnoselskii iteration sequence $\{x_n\} \subset C$ defined by

$$x_{n+1} = \lambda x_n \oplus (1-\lambda)T(x_n), \quad n \geq 1. \tag{3}$$

The following technical lemma is the metric version of the linear case above.

Lemma 4.1 *Let $(X,d,\leq)$ be as above. Let C be a nonempty convex subset of X. Let $T : C \to C$ be a monotone mapping. Assume there exists $x_1 \in C$ such that $x_1 \leq T(x_1)$. Consider the sequence $\{x_n\}$ defined by* (3). *Then $\{x_n\}$ satisfies the following properties*:

(i) $x_n \leq x_{n+1} \leq T(x_n)$, *for any $n \geq 1$*;

(ii) $\lim_{n\to\infty} \|x_n - T(x_n)\| = 0$.

The metric version of Theorem 3.2 is the following.

Theorem 4.1 *Let $(X,d,\leq)$ be as above. Let C be a weakly compact nonempty convex subset of X. Let $T : C \to C$ be a monotone pointwise contraction. Assume there exists $x_1 \in C$ such that $x_1 \leq T(x_1)$. Then T has a fixed point $z \in C$ and $\{T^n(x_1)\}$ converges to z.*

Proof Let $\{x_n\}$ be the sequence generated from x_1 by (3). Since C is weakly compact, $K = \bigcap_{n\geq 1}[x_n, \to) \cap C$ is not empty. Clearly K is convex and $T(K) \subset K$. Define the type function $\tau : K \to [0,+\infty)$ by

$$\tau(x) = \limsup_{n\to\infty} d(x_n, x).$$

Set $\tau_0 = \inf\{\tau(x); x \in K\}$. Obviously τ is a continuous convex function, *i.e.*, the set $\{x; \tau(x) \leq r\}$ is convex for any $r \geq 0$. Since C is weakly compact convex subset of X, we conclude the existence of $z \in C$ such that $z \in K$ and $\tau(z) = \tau_0$. Since T is monotone, we have $T(z) \in K$. Hence $\tau(z) \leq \tau(T(z))$. On the other hand, we have

$$\tau\bigl(T(z)\bigr) = \limsup_{n\to\infty} d\bigl(x_n, T(z)\bigr) = \limsup_{n\to\infty} d\bigl(T(x_n), T(z)\bigr).$$

Since $x_n \leq z$, for any $n \geq 1$, we get

$$d\bigl(T(x_n), T(z)\bigr) \leq k(z) d(x_n, z), \quad n \geq 1,$$

which implies

$$\limsup_{n\to\infty} d\bigl(T(x_n), T(z)\bigr) \leq k(z) \limsup_{n\to\infty} d(x_n, z).$$

Hence $\tau(z) \leq k(z)\tau(z)$. Since $k(z) < 1$, we conclude that $\tau(z) = 0$, which implies

$$\limsup_{n\to\infty} d(x_n, z) = 0,$$

i.e., $\{x_n\}$ converges to z. Next we show that $z \in \mathrm{Fix}(T)$. Since

$$d\bigl(T(x_n), T(z)\bigr) \leq k(z) d(x_n, z), \quad n \geq 1,$$

we conclude that $\{T(x_n)\}$ converges to $T(z)$. Finally $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$ implies that $T(z) = z$. The last part of this theorem follows from the fact that $x_1 \leq z$. $\square$

Note that the conclusion of Theorem 4.1 is still valid if we assume $T(x_1) \leq x_1$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the manuscript.

Author details
[1]Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia. [2]Department of Mathematical Sciences, University of Texas at El Paso, El Paso, TX, USA. [3]Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Dhahran, 31261, Saudi Arabia.

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